Math 275D Lecture 18 Notes

Daniel Raban

November 13, 2019

1 The Itô Integral for L^2 Functions

1.1 The Itô integral for simple functions

Recall: If $f \in L^2(\Omega_B \otimes [0,T])$, where $f(\omega,t)$ is $\mathcal{F}_t$ measurable for each t, we want to understand the integral

$$I_T(f) = \int_0^T f(\omega,t)\, dB_t.$$

If $f(\omega,t)$ is the number of stocks we have at time t, then the integral gives the profit we get between 0 and T. We start by analyzing the function

$$f = a(\omega)\mathbb{1}_{[t_1,t_2]}(t), \qquad 0 \le t_1 \le t_2 \le T$$

This is a step function on a fixed interval with a random height. In this case,

$$I_T(f) = a(\omega) \cdot B(t_2) - B(t_1).$$

This is intuitive: this function says we buy a stocks at time t_1 and sell them at t_2; so the profit is the change in price of stocks from t_1 to t_2 times the number of shares I have.

The integral is linear, so for simple functions,

$$f = \sum_{k=1}^n a_k(\omega)\mathbb{1}_{[t_k^1,t_k^2]}(t) \quad a_k \in \mathcal{F}_{t_k^1} \implies I_T(f) = \sum_{k=1}^n a_k(\omega)(B(t_k^2) - B(t_k^1)).$$

1.2 Extending the Itô integral to general L^2 functions

If $f \in L^2$, we want to find a sequence of simple functions $f_k \xrightarrow{L^2} f$ so we can let $I_T(f) := \lim_k I_T(f_k)$.

Lemma 1.1. *If f is a simple function,*

$$\|f\|_{L^2(\Omega\times[0,T])} = \|I_T(f)\|_{L^2(\Omega)}.$$

Proof. Suppose $f = a(\omega)\mathbb{1}_{[s_1,s_2]}(t) + b(\omega)\mathbb{1}_{[t_1,t_2]}$ with $s_1 < t_1 < s_2 < t_2$. First,

$$\mathbb{E}\left[\int_0^T f^2\,dt\right] = \mathbb{E}[(t_1 - s_1)a^2 + (s_2 - t_1)(a+b)^2 + (t_2 - s_2)b^2].$$

On the other hand,

$$\mathbb{E}[I_T^2(f)] = \mathbb{E}[[(B(t_1) - B(s_1))a + (B(s_2) - B(t_1))(a+b) + (B(t_2) - B(s_2))b]^2]$$

Say the intervals are $J_1 = [s_1, t_1]$, $J_2 = [t_1, s_2]$ and $J_3 = [s_2, t_2]$. Then if we look at $\mathbb{E}[B(J_1)aB(J_2)(a+b)]$ for example, $B(J_2)$ is independent of the rest. So the crossing terms cancel.

$$\begin{aligned} &= \mathbb{E}[(B(t_1) - B(s_1))^2 a^2 + (B(s_2) - B(t_1))^2 (a+b)^2 + (B(t_2) - B(s_2))^2 b^2] \\ &= (t_1 - s_1)\,\mathbb{E}[a^2] + (s_2 - t_1)\,\mathbb{E}[(a+b)^2] + (t_2 - s_2)\,\mathbb{E}[b^2]. \end{aligned}$$ $\square$

So we get that $I_T : L^2(\Omega \times [0,1]) \to L^2(\Omega)$ is isometric on simple functions. So if $\|f_m - f_n\|_2 \to 0$, we get $\|I_T(f_m) - I_T(f_n)\|_2 \to 0$. So we can convert Cauchy sequences in $L^2(\Omega \times [0,1])$ to Cauchy sequences in $L^2(\Omega)$, find the limit, and use it to define $I_T(f)$.

Remark 1.1. $I_T(f)$ is "only" L^2-unique. So if $h = I_T(f)$ except in a probability 0 set, h is also $\lim I_T(f_k)$. This is the same as with the definition of conditional expectation.

1.3 The Itô integral as a random function

If $t \le T$, let

$$F(\omega, t) = \int_0^t f(\omega, s)\,dB_s$$

F is a random function. We should believe that $F \in C([0,1])$ a.s. Here is the issue: for any t, we know $F(\omega,t)$ in a probability 1 set. But we want to have a random variable for all t at once. We can have $\mathbb{P}(\widetilde{F}(\omega,t) = F(\omega,t)) = 1$ for fixed t, but it is still possible that $\mathbb{P}(\widetilde{F} \neq F) = 1$ because $\{\widetilde{F} = F\} = \bigcap_t \{\widetilde{F}(\omega,t) = F(\omega,t)\}$; this is an uncountable intersection.

So if we want to define the random function $F(\omega,t)$, it is not a "simple" extension of $I_T(f)$. To make the construction work out, we need to make sure F is continuous. But this is hard in general; in general, if I have X_t for each t, it's not easy to find $F(t) \in C([0,1])$ with $F(t) \overset{d}{=} X_t$ for each t. We will need to find a sequence of continuous functions that converge uniformly to $F(\omega,t)$.